NUMERICAL COMPARISON OF SOME ESTIMATORS OF VARIANCE UNDER PPS SYSTEMATIC SAMPLING

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# I. Introduction

The purpose of this study is to investigate the performance of several estimators of the variance of the Horvitz-Thompson (HT) estimator of total, $\hat{Y}_{HT} = \sum_{i=1}^{n} y_i/\pi_i$, under a probability proportional to size (PPS) systematic sampling design. The PPS systematic sampling scheme was selected for study because of its wide applicability and usage. Since the PPS systematic scheme does not yield an unbiased estimator of variance, a comparative study of the biases and mean square errors (MSE's) of several variance estimators in a real finite population was conducted.

The population used in the study consisted of mobile home dealers canvassed in the 1972 Census of Retail Trade. The estimators of variance chosen for study include those most commonly found in the literature plus some minor variations. We considered two variables, referred to as y and z, as characteristics to be estimated. All of the results were obtained for two universes distinguished by two different orderings of the population of mobile home dealers. We refer to the population ordered by decreasing measure of size as Universe I and to a second ordering of the population (roughly a geographical ordering of the units) as Universe II. Given the results from the two universes, the effect that the order of the units in the frame has upon the variance of the estimator of total and on the estimation of variance is considered.

# II. Description of the Study

## 1. Preparation

The population used in the study consisted of a compact file of mobile home dealers canvassed in the 1972 Census of Retail Trade. The data record for each mobile home dealer contained an identification number, 1972 annual sales, 1972 average quarterly payroll and 1972 first-quarter employment. Universe II was obtained by sorting on the identification number. For the purposes of this study, the 19 largest mobile home dealers were excluded on the basis of their size (these units would be designated as certainty units in most sample designs), and a few of the very smallest dealers (in terms of payroll) were excluded to simplify the computer programming. We considered 1972 annual sales (y) and 1972 first-quarter employment (z) as characteristics to be estimated and 1972 average quarterly payroll (x) was used as a measure of size, i.e., $p_i = x_i/X$ and $X = \sum_{i=1}^{N} x_i$. The payroll figures for a few of the dealers were adjusted slightly so that X was divisible by the sample sizes (n=30, 60, 150, 300) considered in the study.

## 2. Estimators of Variance

The following variance estimators were utilized for estimating the variance of $\hat{Y}_{HT}$ and the variance of $\hat{Z}_{HT} = \sum_{i=1}^{n} z_i/\pi_i$ over all possible systematic samples of sizes n=30, 60, 150, and 300 from each of the two universes:

a. Random group estimator with t groups -

$$rg(t) = 1/t \sum_{g=1}^{t} \frac{(y_g-\bar{y})^2}{t-1} \qquad t=5,10,15,20,30$$

b. With replacement variance estimator -

$$WR = \sum_{i=1}^{n} \frac{\left(\frac{y_i}{p_i} - \hat{Y}_{HT}\right)^2}{n(n-1)}$$

c. With replacement variance estimator with adjustment -

$$WRA = \left[1 - \sum_{i=1}^{n} \frac{\pi_i}{n}\right] \times WR$$

d. "Randomized systematic" variance estimator-

$$RRS = \frac{1}{n-1} \sum_{i<i'}^{n} \left[1-(\pi_i + \pi_{i'}) + \sum_{k=1}^{N} \frac{\pi_k^2}{n}\right]\left(\frac{y_i}{\pi_i} - \frac{y_{i'}}{\pi_{i'}}\right)^2$$

e. Collapsed stratum variance estimator -

$$CS = \sum_{h=1}^{n/2} \left(\frac{y_i}{np_i} - \frac{y_j}{np_j}\right)_h^2$$

where i and j are adjacent pairs of units in the sample.

f. Successive pairs variance estimator -

$$SP = \frac{n}{2(n-1)} \sum_{i=1}^{n-1} \left(\frac{y_i}{np_i} - \frac{y_{i+1}}{np_{i+1}}\right)^2$$

g. Successive pairs variance estimator with adjustment -

$$SPW = \left[1 - \sum_{i=1}^{n} \frac{\pi_i}{n}\right] \times SP$$

All of the variance estimators specified above can either be found in the literature or in the usual sampling texts. The RRS estimator was proposed [4,7] in the context of PPS systematic sampling when the units in the population to be sampled from are randomly arranged in one of the N! possible sequences. Although the population under study was placed separately in two specified orders, it was felt that it would be of interest to include RRS in the comparison. The CS and SP estimators were felt to be reasonable estimators of the variance under a PPS systematic design when one visualized the actual sample design as being approximated by a one sample unit per stratum design where the strata consist of units lying within the realized sampling intervals. The WR estimator is a special case of rg(t) when n=t. It can be shown that WR has the same bias of rg(t) but a smaller MSE than rg(t).

In addition to the estimators listed above, another estimator which we call the pseudo random group (prg) estimator was considered. Estimators prg(t) and rg(t) have the same form, but they differ in the manner in which the sample units are assigned to the t groups. In rg(t) the sample units are assigned randomly to the t groups while in prg(t) the sample units are assigned to groups systematically in the order which they are selected into the sample.

$\hat{Y}_{HT}$ and $\hat{Z}_{HT}$ were calculated for every possible sample of a given size, the samples of units not necessarily being unique, and $V(\hat{Y}_{HT})$ and $V(\hat{Z}_{HT})$ were calculated for each sample size in each universe. The expected value of each of the estimators of $V(\hat{Y}_{HT})$ and $V(\hat{Z}_{HT})$ (except rg(t)) was obtained by averaging the estimates over all possible samples of the given size. The variance of each of these variance estimators was also calculated.

The mean and variance of rg(t) were calculated in the following manner.

i. Using the result referred to earlier, we set

$$E[rg(t)] = E[WR]$$

ii. It can be shown that

$$Var[rg] = E\{Var[rg|sample]\} + Var[WR].$$

Hence Var[rg|sample] was calculated for each sample and averaged over all samples. This term was then added to Var[WR].

Having obtained the mean and variance of each estimator, we calculated the mean square error of each. The results of these calculations are provided in Tables 1 and 2.

Further distributional properties of the estimators of $V(\hat{Y}_{HT})$ are reflected by the confidence interval results in Table 3. These proportions were obtained in the following manner. For a given sample, 90 and 95% confidence intervals were constructed for Y (and Z) using the $\hat{Y}_{HT}$ ($\hat{Z}_{HT}$) estimate and each of the estimates of ($\hat{V}_{HT}$)($V(\hat{Z}_{HT})$) produced by that sample. For example, for 95% confidence intervals, $\hat{Y}_{HT} \pm 1.96 \sqrt{SPY}$ was calculated for each sample where SPY = the SP estimator of $V(\hat{Y}_{HT})$ for the given sample. 90 and 95% confidence intervals using $\hat{Y}_{HT}(\hat{Z}_{HT})$ and its variance for each possible PPS systematic sample were also constructed (e.g., for 95% confidence intervals, $\hat{Y}_{HT} \pm 1.96 \sqrt{V(\hat{Y}_{HT})}$).

Then, for each estimator, the true proportion of the confidence intervals which contained Y(Z) was calculated as was the proportion of confidence intervals constructed using the variance of $\hat{Y}_{HT}$ ($\hat{Z}_{HT}$). These calculations were made for each sample size and universe. The proportions are provided in Table 3.

3. Summary Parameters

The intraclass correlation, $\rho$, was calculated for each sample size and is shown in Table 7 along with its lower bound, $-(\frac{1}{n-1})$. $\rho$ is defined in [3] where it is shown to be expressible alternatively as

$$\rho_Y = \frac{1}{n-1}\,[V(\hat{Y}_{HT}) - V(\hat{Y}')]/V(\hat{Y}')$$

where $V(\hat{Y}')$ is the variance of the estimator $\hat{Y}' = \sum_{i=1}^{n} y_i/np_i$ , under with replacement PPS sampling; that is,

$$V(\hat{Y}') = \frac{1}{n}\sum_{i=1}^{N} p_i\left(\frac{y_i}{p_i} - Y\right)^2 .$$

The term referred to as VS in Table 7 is an approximation to the variance of $\hat{Y}_{HT}$ under a sample design in which the units in the population are randomly ordered and a PPS systematic design is used to select the sample of n units [4,7]. It has the following form:

$$VS = \sum_{i=1}^{N} \pi_i\left[1 - \frac{n-1}{n}\pi_i\right]\left(\frac{y_i}{\pi_i} - \frac{Y}{n}\right)^2 .$$

Its magnitude, relative to $V(\hat{Y}_{HT})$, is presented in Table 7. The remaining columns in Table 7 provide ratios of the variances resulting from several alternative estimator-sample design pairs relative to $V(\hat{Y}_{HT})$.

III. Results

1. Estimators of $V(\hat{Y}_{HT})$ and $V(\hat{Z}_{HT})$

For the population of N=5634 mobile home dealers,

$$Y = \sum_{i=1}^{N} y_i = .32385 \times 10^7$$

$$Z = \sum_{i=1}^{N} z_i = .33213 \times 10^5$$

$$\text{and } X = \sum_{i=1}^{N} x_i = .57300 \times 10^5$$

Graphs 1 and 2 illustrate the relationships between the variables y and x and between z and x in the population. The plots indicated that x would be a useful design variable. The correlation coefficients squared are, respectively, .74 and .75.

Tables 1A and 1B present, for Universe I, the expected values and MSE's (relative to MSE(WR)) of the estimators discussed in Section II.2. The MSE's of rg(•) for n=150 and 300 were not calculated due to limited resources and because it was observed that the MSE's for rg(•) for n=30 and 60 did not differ appreciably from the MSE's for prg(•). In the case of rg(•), when the sample size was such that the random groups did not contain equal numbers of sample units, the MSE was not calculated. Tables 2.A and 2.B are similar to Tables 1.A and 1.B except that the results refer to Universe II. In the following all conclusions and summaries refer solely to the mobile home dealer population under study.

In terms of relative bias, CS had the smallest bias in the largest number of the 8 characteristic/sample size combinations in Universe I and appeared to possess a bias slightly larger than that of the smallest in the other cases. In Universe II, WR had the smallest relative bias for the y characteristic while for the z characteristic, no estimator stood out.

With respect to MSE, SPW consistently exhibited the smallest MSE in Universe I. Other estimators with reasonably small MSE's were SP, CS, and WRA. For Universe II, RRS, WRA, CS, pg (15) and SPW had the smallest MSE for at least one characteristic/sample size combination with RRS appearing best overall. In general, the estimators rg(•) and prg(•) performed poorest of all over the 16 cases with prg(•) performing better than rg(•).

One interesting observation can be made with respect to WR and RRS and the relative bias. That is, for a given characteristic/sample size combination each of the estimators exhibit, approximately, the same expected value whether applied in Universe I or II. When $\rho_y(\rho_z)$ is negative and hence PPS systematic sampling is superior to PPS with replacement sampling, the relative biases of WR and RRS are positive and vice versa when $\rho_y(\rho_z)$ is positive (except for one case). This result probably occurs because WR and RRS do not reflect the systematic nature of the sampling design as compared to CS and SP. Hence, when $\rho$ is negative, and the ratios $y_i/\pi_i$ in the sample are diverse, WR estimates too high, and when $\rho$ is positive the ratios in the sample are similar and hence WR estimates too low.

The results of the confidence interval calculations are located in Table 3. The proportions obtained from intervals constructed using $\hat{Y}_{HT}(\hat{Z}_{HT})$ and $V(\hat{Y}_{HT})(V(\hat{Z}_{HT}))$ are, in most of the 16 universe/characteristic/sample size combinations, greater than the .90 (or .95) which would have been expected from a normally distributed $\hat{Y}_{HT}(\hat{Z}_{HT})$. In those cases in which the proportions did not exceed .90 (or .95) they were very close.

A few general comments may be made concerning the proportions resulting from the confidence intervals constructed with $\hat{Y}_{HT}(\hat{Z}_{HT})$ and the estimates of $V(\hat{Z}_{HT})$). As the sample size increased the number of individual proportions which exceeded the .90 (or .95) levels rose. Over both universes, and for sample sizes n=30,60,150,300 the number of proportions greater than .90 (or .95) equaled 10,14,36, and 29, respectively (out of 88). Of these, 9,14,26, and 22 were in Universe I. Very few proportions from Universe II ever reached the .90 (or .95) levels.

In terms of the performances of the individual variance estimators in producing their associated proportions, prg20, prg30, WR, WRA and RRS produced the highest proportions in nearly every universe/characteristic/sample size combination. It was the proportions resulting from these estimators which most often exceeded the .90 (or .95) levels.

3. Alternative Estimator-Sample Design Pairs

Table 7 illustrates the $\rho$ value for each universe/characteristic/sample size combination along with the variances of seven other estimator-sample design pairs. The column headed $V_{SYS}[\cdot]$ represents the variance of $\hat{Y}_{HT}(\hat{Z}_{HT})$ under PPS systematic sampling using quarterly payroll as the measure of size. $V_{WR}[\cdot]$ repre- the variance of $\hat{Y}' = \sum_{i=1}^{n} \frac{y_i}{np_i}$ $(\hat{Z}' = \sum_{i=1}^{n} \frac{z_i}{np_i})$ under PPS with replacement sampling. $V_{SRS}[\cdot]$ represents the variance of the HT estimator of total under an SRS without replacement design, and $V_{SRS}$[Ratio, X] denotes the variance of the ratio estimator using X = quarterly payroll as the auxiliary variable under an SRS without replacement design. $V_{SYS}$[Ratio, Z] refers to the variance of the ratio estimator of total using Z as the auxiliary variable under a PPS systematic design. $V_{WR}$[Ratio, Z] represents the variance of the ratio estimator of total using Z as the auxiliary variable under a PPS with replacement design. $V_{SRS}$[Ratio, Z] is analogous to $V_{SRS}$[Ratio, X] with Z used as the auxiliary variable. The entries in the table express the above-described variances relative to $V_{SYS}(\cdot)$.

The simple raw correlation between Y and Z is $\rho_{Y,Z} = .789$. This implies, since $\rho_{Y,Z} > 1/2 \frac{C.V.(Z)}{C.V.(Y)}$, in the simple random sampling context, that the ratio estimator is preferable. The entries in the table under $V_{SRS}(\cdot)$ and $V_{SRS}$[Ratio, Z] support the choice of the ratio estimator in this situation. However, neither of these estimator-sample design pairs does better than $V_{SYS}(\cdot)$ for any universe/characteristic/sample size combination.

Comparison of $V_{SYS}(\cdot)$ and $V_{SYS}$[Ratio,Z] shows that the HT estimator performs better in 5 or 8 cases, and the ratio estimator does better in the other three cases. In the PPS systematic context, the relevant correlation in deciding between the HT estimator and a ratio estimator is no longer the raw correlation between Y and Z, but is the correlation between $\hat{Y}_{HT}$ and $\hat{Z}_{HT}$, designated as $\rho_{\hat{Y},\hat{Z}}$, and the criterion for selection of the ratio estimator over the HT estimator is

$$\rho_{\hat{Y},\hat{Z}} \quad 1/2 \frac{C.V.(\hat{Z}_{HT})}{C.V.(\hat{Y}_{HT})}$$

In those cases in which the ratio estimator is superior to HT (has smaller variance), even when accounting for the bias of the ratio estimator, it remains better than HT. This follows from the results of Table 4 where it is seen that the MSE of $\hat{Y}_R$ is lower than the variance of $\hat{Y}_{HT}$ in Universe II for n=60,150 and 300. However, in these cases, the estimator-sample design producing $V_{WR}(\cdot)$ performs even better than $V_{SYS}$[Ratio, Z] and, from the table we see that VS is even better than $V_{WR}(\cdot)$ in these cases (VS can be shown to be better than $V_{WR}(\cdot)$ in general).

In almost all cases, when $\rho<0$, $V_{SYS}(\cdot) \leq$ VS $\leq V_{WR}(\cdot)$. From [7], we know that VS $\leq V_{WR}(\cdot)$; hence, when $\rho>0$, VS $\leq V_{WR}(\cdot) \leq V_{SYS}(\cdot)$. From a practical standpoint, if we decide to use PPS systematic sampling and the Horvitz-Thompson estimator and suspect that $\rho<0$, we can use VS as a "safe" (larger than $V_{SYS}(\cdot)$) approximation to $V_{SYS}(\cdot)$ for design purposes.

In general, Table 7 shows that the estimator-sample design pairs resulting in VS or $V_{SYS}(\cdot)$ are better than the rest, and the sign of $\rho$ appears to determine which of the two is preferable. Also, Table 7 demonstrates that gains of at least 20% in $V_{SYS}(\cdot)$ can be realized by using Universe I over Universe II, a consequence of the negative $\rho$ induced by the ordering.

## 4. Conclusion

In practice, one never really knows whether $\rho$ will be negative with respect to the characteristics to be estimated. However, in many instances comparable data is available on the same population for a previous point in time. Graphs 3 and 4 are plots of the ratios of sales to payroll and employment size to payroll, respectively. As is evident in the two graphs, an ordering of the units by size of payroll and a systematic sampling scheme will tend to spread the ratios evenly over the possible samples and hence make the ratios within the samples diverse, thereby possibly inducing a negative $\rho$. It is speculated that the same analysis performed on other populations with similar graphs as those of the mobile home dealer population will produce results comparable to the variance estimator comparisons arrived at as a result of Tables 1, 2 and 3. Hence, faced with another population of interest with similar graphs as in Graphs 3 and 4, one can use the results concerning the variance estimators of this limited study in the decision-making process.

The tables containing the results on the estimation of Var($\hat{Y}_{HT}$) and Var($\hat{Z}_{HT}$) in Universe II have been omitted due to space limitations. Also, both text and tables relating to the estimation of Var($\hat{Y}_R$), where $\hat{Y}_R=(\hat{Y}_{HT}/\hat{Z}_{HT})$ Z, have been omitted, as have all graphs referred to in the text. Interested readers may contact the authors for these results.

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**TABLE 1 A.** Expected Value and Relative MSE $\left(\text{Rel MSE} = \frac{MSE(\cdot)}{MSE(WR)}\right)$ of Some Variance Estimators of $\hat{Y}_{HT}$ (Universe I)

| | n = 30 | | n = 60 | | n = 150 | | n = 300 | |
|---|---|---|---|---|---|---|---|---|
| | Expected Value $\times 10^{12}$ | Rel MSE | Expected Value $\times 10^{11}$ | Rel MSE | Expected Value $\times 10^{11}$ | Rel MSE | Expected Value $\times 10^{11}$ | Rel MSE |
| prg (5) | .1616 | .8502 | .7125 | .8449 | .2879 | .8823 | .1384 | .8067 |
| prg (10) | .1658 | .9026 | .7970 | .8508 | .3003 | .7168 | .1404 | .8898 |
| prg (15) | .1741 | .9807 | .8096 | .9624 | .3117 | 1.0084 | .1500 | .9518 |
| prg (20) | .2525 | 1.0770 | .8228 | .8998 | .3624 | .9412 | .1482 | .7324 |
| prg (30) | .1815 | 1.0000 | .8650 | .9784 | .3245 | .9597 | .1544 | .9884 |
| WR | .1815 | 1.0000 | .9039 | 1.0000 | .3609 | 1.0000 | .1803 | 1.0000 |
| WRA | .1783 | .9655 | .8718 | .9277 | .3288 | .8164 | .1482 | .6784 |
| RRS | .1803 | .9982 | .8923 | .9943 | .3493 | .9743 | .1687 | .9569 |
| CS | .1468 | .8076 | .7230 | .9701 | .2864 | .7009 | .1424 | .6820 |
| SP | .1224 | .2691 | .6364 | .3228 | .2769 | .7942 | .1417 | .8060 |
| SPW | .1202 | .2603 | .6139 | .2932 | .2522 | .6624 | .1164 | .6146 |
| rg (5) | .1815 | 1.0433 | .9039 | 1.0951 | | | | |
| rg (10) | .1815 | 1.0151 | .9039 | 1.0381 | | | | |
| rg (15) | .1815 | 1.0074 | .9039 | 1.0219 | | | | |
| rg (20) | | | .9039 | 1.0145 | | | | |
| rg (30) | .1815 | 1.0000 | .9039 | 1.0071 | | | | |
| V ($\hat{Y}_{HT}$) | .1428 | | .7177 | | .2762 | | .1642 | |
| MSE (WR) | $.2843 \times 10^{24}$ | | $.3513 \times 10^{23}$ | | $.2260 \times 10^{22}$ | | $.2758 \times 10^{21}$ | |

**TABLE 1 B.** Expected Value and Relative MSE $\left(\text{Rel MSE} = \frac{MSE(\cdot)}{MSE(WR)}\right)$ of Some Variance Estimators of $\hat{Z}_{HT}$ (Universe I)

| | n = 30 | | n = 60 | | n = 150 | | n = 300 | |
|---|---|---|---|---|---|---|---|---|
| | Expected Value $\times 10^{8}$ | Rel MSE | Expected Value $\times 10^{7}$ | Rel MSE | Expected Value $\times 10^{7}$ | Rel MSE | Expected Value $\times 10^{7}$ | Rel MSE |
| prg (5) | .1256 | 1.1964 | .6074 | 1.4989 | .2146 | 2.1343 | .1092 | 1.5865 |
| prg (10) | .1237 | 1.0870 | .6216 | 1.2362 | .2186 | 1.4194 | .1118 | 1.4042 |
| prg (15) | .1207 | 1.0199 | .5883 | 1.0624 | .2304 | 1.3938 | .1094 | 1.0014 |
| prg (20) | .1960 | 1.5726 | .6157 | 1.0982 | .2625 | 1.5744 | .1114 | 1.0059 |
| prg (30) | .1226 | 1.0000 | .6025 | 1.0241 | .2315 | 1.0939 | .1163 | 1.1348 |
| WR | .1226 | 1.0000 | .6122 | 1.0000 | .2444 | 1.0000 | .1223 | 1.0000 |
| WRA | .1204 | .9630 | .5904 | .9179 | .2226 | .8012 | .1005 | .4893 |
| RRS | .1214 | .9947 | .6003 | .9839 | .2330 | .9480 | .1110 | .7708 |
| CS | .1075 | .8996 | .5343 | .8365 | .2086 | .8455 | .1064 | .6500 |
| SP | .0982 | .3514 | .5052 | .3264 | .2041 | .7280 | .1063 | .6483 |
| SPW | .0964 | .3408 | .4872 | .3042 | .1859 | .6442 | .0873 | .3177 |
| rg (5) | .1226 | 1.2452 | .6122 | 1.5423 | | | | |
| rg (10) | .1226 | 1.0870 | .6122 | 1.2166 | | | | |
| rg (15) | .1226 | 1.0460 | .6122 | 1.1249 | | | | |
| rg (20) | | | .6122 | 1.0818 | | | | |
| rg (30) | .1226 | 1.0000 | .6122 | 1.0402 | | | | |
| V ($\hat{Z}_{HT}$) | .1076 | | .5063 | | .2176 | | .0697 | |
| MSE (WR) | $.2459 \times 10^{15}$ | | $.3107 \times 10^{14}$ | | $.2077 \times 10^{13}$ | | $.4948 \times 10^{12}$ | |

Table 3A
Confidence Levels for Intervals Constructed With $\hat{Y}_{HT}$ ($\hat{Z}_{HT}$) and Several Estimators of $V(\hat{Y}_{HT})$ ($V(\hat{Z}_{HT})$) for n = 30

| Variance estimator | Universe I | | | | Universe II | | | |
|---|---|---|---|---|---|---|---|---|
| | Y | | Z | | Y | | Z | |
| | .90 | .95 | .90 | .95 | .90 | .95 | .90 | .95 |
| prg5 | .8408 | .8984 | .8539 | .9089 | .7723 | .8278 | .7937 | .8482 |
| prg10 | .8691 | .9215 | .8890 | .9398 | .8958 | .8644 | .8372 | .8984 |
| prg15 | .8775 | .9298 | .8901 | .9387 | .8288 | .8712 | .8534 | .9026 |
| prg20 | .9555 | .9817 | .9660 | .9859 | .8770 | .9215 | .9387 | .9670 |
| prg30 | .8901 | .9330 | .9026 | .9482 | .8346 | .8791 | .8524 | .9094 |
| WR | .8901 | .9330 | .9026 | .9482 | .8346 | .8791 | .8524 | .9094 |
| WRA | .8885 | .9288 | .9000 | .9456 | .8319 | .8775 | .8482 | .9047 |
| RRS | .8885 | .9298 | .9000 | .9461 | .8330 | .8775 | .8513 | .9063 |
| CS | .8503 | .9016 | .8707 | .9330 | .8152 | .8681 | .8382 | .8974 |
| SP | .8461 | .9042 | .8607 | .9319 | .8220 | .8733 | .8450 | .8995 |
| SPW | .8429 | .9011 | .8586 | .9309 | .8178 | .8728 | .8429 | .8979 |
| $V(\hat{Y}_{HT})$ ($V(\hat{Z}_{HT})$) | .9319 | .9607 | .9115 | .9529 | .9236 | .9602 | .9052 | .9482 |

Table 3B
Confidence Levels for Intervals Constructed With $\hat{Y}_{HT}$ ($\hat{Z}_{HT}$) and Several Estimators of $V(\hat{Y}_{HT})$ ($V(\hat{Z}_{HT})$) for n = 60

| Variance estimator | Universe I | | | | Universe II | | | |
|---|---|---|---|---|---|---|---|---|
| | Y | | Z | | Y | | Z | |
| | .90 | .95 | .90 | .95 | .90 | .95 | .90 | .95 |
| prg5 | .8189 | .8754 | .8545 | .9141 | .7874 | .8398 | .8461 | .8848 |
| prg10 | .8806 | .9194 | .8963 | .9435 | .8335 | .8838 | .8660 | .9141 |
| prg15 | .8890 | .9340 | .8869 | .9414 | .8524 | .9100 | .8744 | .9298 |
| prg20 | .8932 | .9382 | .9058 | .9623 | .8513 | .9110 | .8744 | .9183 |
| prg30 | .9026 | .9435 | .9058 | .9529 | .8702 | .9257 | .8932 | .9351 |
| WR | .9110 | .9487 | .9100 | .9602 | .8691 | .9309 | .8932 | .9298 |
| WRA | .9026 | .9476 | .9068 | .9550 | .8649 | .9309 | .8901 | .9278 |
| RRS | .9068 | .9476 | .9079 | .9571 | .8681 | .9309 | .8911 | .9278 |
| CS | .8691 | .9152 | .8848 | .9319 | .8628 | .9236 | .8806 | .9278 |
| SP | .8586 | .9194 | .8848 | .9351 | .8754 | .9225 | .8869 | .9319 |
| SPW | .8524 | .9131 | .8817 | .9298 | .8639 | .9194 | .8838 | .9246 |
| $V(\hat{Y}_{HT})$ ($V(\hat{Z}_{HT})$) | .9215 | .9529 | .8995 | .9508 | .9215 | .9508 | .8995 | .9550 |

Table 3C

Confidence Levels for Intervals Constructed With $\hat{Y}_{HT}$ ($\hat{Z}_{HT}$) and Several Estimators of $V(\hat{Y}_{HT})$ ($V(\hat{Z}_{HT})$) for $n = 150$

| Variance estimator | Universe I | | | | Universe II | | | |
|---|---|---|---|---|---|---|---|---|
| | Y | | Z | | Y | | Z | |
| | .90 | .95 | .90 | .95 | .90 | .95 | .90 | .95 |
| prg5 | .8508 | .8822 | .8272 | .8979 | .8272 | .8874 | .8115 | .8560 |
| prg10 | .8874 | .9189 | .8796 | .9162 | .8246 | .9058 | .8534 | .9110 |
| prg15 | .9031 | .9372 | .9136 | .9476 | .8325 | .8953 | .8456 | .9084 |
| prg20 | .9136 | .9555 | .9189 | .9555 | .8639 | .9136 | .8639 | .9215 |
| prg30 | .9136 | .9529 | .9241 | .9581 | .8351 | .9162 | .8456 | .9058 |
| WR | .9267 | .9634 | .9346 | .9738 | .8377 | .9189 | .8586 | .9084 |
| WRA | .8189 | .9529 | .9189 | .9686 | .8089 | .8874 | .8456 | .8822 |
| RRS | .9215 | .9634 | .9189 | .9634 | .8272 | .9084 | .8403 | .9031 |
| CS | .8848 | .9476 | .9031 | .9607 | .8429 | .9005 | .8508 | .9005 |
| SP | .8927 | .9450 | .9136 | .9581 | .8482 | .9031 | .8560 | .9005 |
| SPW | .8639 | .9319 | .8901 | .9424 | .8063 | .8901 | .8377 | .8874 |
| $V(\hat{Y}_{HT})$ ($V(\hat{Z}_{HT})$) | .9267 | .9581 | .9162 | .9529 | .9110 | .9607 | .8979 | .9503 |

Table 3D

Confidence Levels for Intervals Constructed With $\hat{Y}_{HT}$ ($\hat{Z}_{HT}$) and Several Estimators of $V(\hat{Y}_{HT})$ ($V(\hat{Z}_{HT})$) for $n = 300$

| Variance estimator | Universe I | | | | Universe II | | | |
|---|---|---|---|---|---|---|---|---|
| | Y | | Z | | Y | | Z | |
| | .90 | .95 | .90 | .95 | .90 | .95 | .90 | .95 |
| prg5 | .7801 | .8534 | .8901 | .9162 | .7120 | .7749 | .8377 | .9215 |
| prg10 | .8325 | .9005 | .9215 | .9529 | .7435 | .8325 | .8953 | .9476 |
| prg15 | .8744 | .9215 | .9581 | .9895 | .7644 | .8534 | .8796 | .9267 |
| prg20 | .8744 | .9319 | .9424 | .9738 | .8011 | .8796 | .9110 | .9476 |
| prg30 | .8848 | .9424 | .9686 | .9895 | .7853 | .8691 | .9058 | .9634 |
| WR | .9162 | .9529 | .9791 | .9895 | .8011 | .8744 | .9100 | .9581 |
| WRA | .8796 | .9319 | .9581 | .9843 | .7382 | .8325 | .8691 | .9267 |
| RRS | .8796 | .9424 | .9581 | .9895 | .7539 | .8325 | .8796 | .9424 |
| CS | .8639 | .9319 | .9529 | .9895 | .7906 | .8586 | .8901 | .9581 |
| SP | .8744 | .9215 | .9581 | .9895 | .7906 | .8639 | .8901 | .9581 |
| SPW | .8272 | .9058 | .9372 | .9943 | .7278 | .8272 | .8586 | .9372 |
| $V(\hat{Y}_{HT})$ ($V(\hat{Z}_{HT})$) | .9162 | .9634 | .9162 | .9529 | .9058 | .9581 | .9058 | .9424 |

Table 7

Universe Parameter Studies (Ratios to $V_{SYS}(\cdot)$)

| | | N | $\rho$ | $-(\frac{1}{n-1})$ | $V_{SYS}(\cdot)$ | $V_{WR}(\cdot)$ | VS | $V_{SRS}(\cdot)$ | $V_{SRS}$ [Ratio,X] | $V_{SYS}$ [Ratio,X] | $V_{WR}$ [Ratio,Z] | $V_{SRS}$ [Ratio,Z] |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Universe I | Y | 30 | −.00715 | −.03448 | $.14275 \times 10^{12}$ | 1.2620 | 1.2542 | 3.9321 | 1.4967 | 1.4262 | 1.4864 | 1.9820 |
| | | 60 | −.00345 | −.01695 | $.71766 \times 10^{11}$ | 1.2551 | 1.2395 | 3.8899 | 1.4805 | 1.3910 | 1.4783 | 1.9606 |
| | | 150 | −.00157 | −.00671 | $.27620 \times 10^{11}$ | 1.3045 | 1.2634 | 3.9779 | 1.5139 | 1.3055 | 1.5365 | 2.0049 |
| | | 300 | −.00030 | −.00334 | $.16424 \times 10^{11}$ | 1.0969 | 1.0275 | 3.2535 | 1.2382 | 1.0219 | 1.2919 | 1.6397 |
| | Z | 30 | −.00409 | −.03448 | $.10758 \times 10^{08}$ | 1.1348 | 1.1243 | 7.2438 | 2.0244 | | | |
| | | 60 | −.00289 | −.01695 | $.50625 \times 10^{07}$ | 1.2059 | 1.1831 | 7.6555 | 2.1395 | | | |
| | | 150 | −.00073 | −.00671 | $.21760 \times 10^{07}$ | 1.1221 | 1.0688 | 7.0097 | 1.9588 | | | |
| | | 300 | −.00144 | −.00334 | $.69666 \times 10^{06}$ | 1.7524 | 1.5856 | 10.6490 | 2.9753 | | | |
| Universe II | Y | 30 | .00143 | −.03448 | $.18764 \times 10^{12}$ | .9601 | .9542 | 2.9914 | 1.1386 | 1.0927 | 1.1308 | 1.5078 |
| | | 60 | .00057 | −.01695 | $.93099 \times 10^{11}$ | .9675 | .9555 | 2.9985 | 1.1413 | .9685 | 1.1396 | 1.5114 |
| | | 150 | .00095 | −.00671 | $.41117 \times 10^{11}$ | .8763 | .8487 | 2.6721 | 1.0170 | .8858 | 1.0321 | 1.3467 |
| | | 300 | .00159 | −.00334 | $.26584 \times 10^{11}$ | .6777 | .6348 | 2.0101 | .7650 | .7810 | .7982 | 1.0130 |
| | Z | 30 | .00139 | −.03448 | $.12700 \times 10^{08}$ | .9613 | .9524 | 6.1361 | 1.7148 | | | |
| | | 60 | −.00096 | −.01695 | $.57591 \times 10^{07}$ | 1.0600 | 1.0400 | 6.7295 | 1.8807 | | | |
| | | 150 | .00058 | −.00671 | $.26534 \times 10^{07}$ | .9202 | .8765 | 5.7485 | 1.6064 | | | |
| | | 300 | −.00019 | −.00334 | $.11505 \times 10^{07}$ | 1.0611 | .9601 | 6.4482 | 1.8017 | | | |